Abstract

The Italian Government finances its public deficit by issuing debt securities, as of October 2005 they account for 86% of public debt. The efficiency of the issuance mechanism of Government Securities is crucial and it entails a correct pricing of the issued bond and notes. The aim of this paper is to address a particular feature of the Italian auction placement mechanism, that has received scant attention by the academic literature. Modeling the short term interest rate as a square root process and I provide a pricing of the option in the framework of the Cox – Ingersoll – Ross model. It turns out that the option helps explaining the mispricing occurring between the primary and secondary market.
Introduction

One of the most important functions of the Italian Treasury is to provide the financing of the public debt at the lowest possible cost for a given level of risk, making a correct pricing of GS extremely important. To this aim, the efficiency of the issuance mechanism of GS is crucial, giving great prominence to the study of market conditions, security design and placement techniques. The present research particularly refers to the third issue.

The Italian GS primary market avails itself of a primary dealership system. The Ministry of Economy and Finance (MEF) chooses a group of intermediaries, called Specialists\(^1\), among the Primary Dealers of the MTS. The status of Specialist implies obligations and privileges, e.g. among the former, Specialists should buy at least 3% of the auction, on the other hand, among the latter, they have exclusive access to the reserved reopenings. Thus, the relationship between the issuer and the primary dealers should be taken into account when considering primary and secondary markets prices. A visible effect of such a relationship on prices is the so-called mispricing, i.e. the prices at which Treasury notes, bills and bonds are sold in auction are higher (overpricing, see Brandolini 2004) or lower (underpricing, see Drudi and Massa 1997, Scalia 1997 and Pacini 2005,) than the when-issued or secondary market prices.

Since Specialists get almost the entire auction, the sign of mispricing presumably depends on the above-mentioned obligations and privileges. To this aim, the quantitative measurement of such obligations and privileges, in terms of cost and profits, is necessary in order to make a complete assessment of the whole placement mechanism. The option implicit in the reopenings reserved to the Specialists is one of the main privileges they have. Indeed, the reopening allows Specialists to buy predetermined additional quantities of GS at the price settled at the auction. The application deadline is fixed at 15.30 p.m. of the business day following the auction. The goal of this research is to provide a pricing of such an option as a call written by MEF with a strike price equal to the stop-out price of the auction. Moreover, in recent years Specialists complain about the existence of overpricing that generates money losses in their balance sheet, a part of the explication is in their aggressive behaviour during auction in order to obtain the right to participate to other special issues.\(^2\) This study could be able to explain another part of the overpricing as the cost of the option embedded in the

\(^1\) D.P.R. n. 398, 30/12/2003.

\(^2\) MEF ranks Specialists according to the quantity they are awarded in the auctions. The higher ranked have the privilege to participate in other particular and highly remunerative operations on the primary market.
reopening procedures. The option is written both on medium and long term bonds and on 6-months BOTs, while the former securities are auctioned through a marginal auction, the latter entail a competitive bidding. This research is focused on medium and long term bonds, hence on BTPs (Buoni Poliennali del Tesoro) and CTZs (Certificati del Tesoro Zero Coupon). The analysis covers the quarters 2004:1-2006:1. The source of the data is the database MTS Time Series.

The rest of the paper is organized as follows: Section I is aimed to determine the characteristics of the implicit option and then to chose a pricing model; in Section II the model is calibrated to data and the option priced; Section III relates the results to the mispricing phenomenon; Section IV concludes.

Section I

In the mid 90’s the reserved reopening was introduced as a privilege for the Specialists participating the ordinary auction. Since its introduction, the feature that has been changing in the years is the time of expiration. For instance, before June 27th 2000 the deadline for bidding at the ordinary auction, and then the option’s starting time, was at 1.00pm. Moreover, since January 15th 2005 the deadline to submitting bids for the reopening changed from 12.00pm to 3.30pm. The higher panel of Table 1.1 contains the number of reopenings and the option’s exercise rate from July 2003 to October 2006 for a selected kind of securities. The lower panel breaks down the sample according to the last change on the option’s expiration. Total exercise means that the complete amount offered in the reserved reopening has been allotted to the Specialists. In the higher panel this quantity goes from about 23% of the CCT to the 60% of the 10 year BTP. The partial exercise is always smaller and goes from 10% in the 10 year BTP to about 30% in the CTZ. The change in the option’s expiration seems to have not significantly affected the rate of exercise.

Table 1.1 Options’ exercise rate by security
<table>
<thead>
<tr>
<th>7/14/03 - 10/31/06</th>
<th># reopenings</th>
<th>total exercise</th>
<th>partial exercise</th>
<th>tot</th>
</tr>
</thead>
<tbody>
<tr>
<td>CTZ</td>
<td>36</td>
<td>47.22%</td>
<td>29.41%</td>
<td>76.63%</td>
</tr>
<tr>
<td>CCT*</td>
<td>13</td>
<td>23.08%</td>
<td>33.33%</td>
<td>56.41%</td>
</tr>
<tr>
<td>BTP 3y</td>
<td>32</td>
<td>46.88%</td>
<td>26.67%</td>
<td>73.54%</td>
</tr>
<tr>
<td>BTP 5y</td>
<td>32</td>
<td>46.88%</td>
<td>13.33%</td>
<td>60.21%</td>
</tr>
<tr>
<td>BTP 10y</td>
<td>33</td>
<td>60.61%</td>
<td>10.00%</td>
<td>70.61%</td>
</tr>
<tr>
<td>total</td>
<td>146</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* 12/29/04 - 2/16/06

<table>
<thead>
<tr>
<th>before 1/15/05</th>
<th># reopenings</th>
<th>total exercise</th>
<th>partial exercise</th>
<th>tot</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>53</td>
<td>43.40%</td>
<td>21.74%</td>
<td>65.14%</td>
</tr>
<tr>
<td>after 1/15/05</td>
<td>93</td>
<td>44.09%</td>
<td>19.51%</td>
<td>63.60%</td>
</tr>
<tr>
<td>total</td>
<td>146</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Overall the participation in reserved reopenings is always above 70% in the most traded securities. Add table-quantity

The first task to address in order to understand the features of the option is to focus on the reopening mechanism. In particular, we have to figure out the option type, the strike price, the expiration date, whether it is European or American and the quantity of the underlying asset. The reopening allows Specialists to buy a predetermined quantity of the auctioned bond at the price settled at the auction. By definition we can describe the option as a plain vanilla call option where the strike price should equal the stop-out price. Anyway, we have to consider that the marginal price includes a placement fee that the Italian Ministry of Finance pays to the Bank of Italy. The Specialists are reimbursed since they can not apply subscription fees to their final clients. The fee is equal to twenty basis points for the 10 year-BTP, thirty for the 3-year BTP and forty for the other maturities. Then the strike price of the option is simply the difference between the marginal price and this fee. The option’s life starts at 11.00am of the auction day and it expires at 3.30pm of the following business day. Hence this is a one-day option. Even if, in principle, each Specialist is able to exercise the option whenever, just short selling the underlying and then placing a bid for the reopening, I model the option as a European one. In fact, the exercise is convenient as soon as the price on the secondary market for the underlying asset exceeds the strike price. Anyway, this approximation is not severe since the short life of the option. The settlement day is the same both for the first auction and for the supplementary placements according to the calendar set at the beginning of the year.
This is of particular interest when considering the strategies that can be implemented by the Specialists using the quantity they are entitled when participating at the reopenings. The reopenings are set up for a maximum amount equal to 25% of the amount offered in the first tranche of every new securities and to 10% for the following placements. Each Specialist has the right to a minimum share of the total amount issued. This fraction equals the sum of the quantities awarded in the last three auctions for the same security, excluding the reopenings, divided by the quantity allotted to all the Specialists in the same auctions. Only those Specialists who took part in the first auction are allowed to participate to the reserved reopenings. Bids are satisfied by first assigning to each Specialist the lesser between the amount requested by the specialist and the amount rightfully due to him. Should one or more Specialists present bids inferior to those rightfully due\(^3\), or not present any bid at all, the difference is allocated to dealers who presented bids greater than those rightfully due. This explains why, in general, it is optimal for the Specialists to bid for a quantity above the minimum one. Now we know the main option’s characteristics and we can turn to the choice of the pricing model.

The tools developed for derivative asset pricing are an example of theories and methods originally developed by physicists in order to solve problems in economics, usually those including uncertainties or stochastic elements and nonlinear dynamics. The option is implicitly written on bonds, hence the spot rate dynamics are the source of uncertainty. These dynamics may be modeled like the ones of a particle in a fluid and then using stochastic differential equations of the kind

\[ dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dw_t, \]

where \(w_t\) is a standard Brownian motion. Given the choice of the drift and the diffusion coefficients a model for the spot rate dynamics can be worked out.

So far only Brandolini (2004) attempted to price the option with the Black and Scholes formula. Although for some classes of derivatives B&S’s assumption of constant risk free rate can be maintained, for instance for options on stock, in general this assumption may not be correct. In particular, when dealing with interest rate derivatives, movements in interest rate are the motivation for the existence of such instruments. Bond options give to the holder the right to buy (call options) or sell (put options) a bond \(P_t\) at a fixed strike price \(K\). Since the

\(^3\) Ex art. 12 of the Issuance Decree. See for instance http://www.dt.tesoro.it/ENGLISH-VE/Public-Deb/Italian-Go/Medium-Lon/index.htm.
bond price depends on the current and future spot rates, bond options will be sensitive to
movements in the underlying interest rate, say r_t. Assuming a constant interest rate would mean
that P_t is completely predictable and as a consequence its volatility should equal zero, if this is
the case there would have been no reasoning for bond options. Since we depart from the B&S
environment the analysis can turn on the choice of an alternative pricing model. The selection
of a proper model involves a number of considerations. With respect to the arbitrage
condition we adopt for the pricing, there are two approaches that can be followed, the
classical and the Heat Jarrow and Morton (1992) ones. In the classical approach to fixed
income a risk-adjusted model for spot rate under the risk neutral martingale measure is
worked out from the price of arbitrage-free bonds. This approach requires to estimate drift and
volatilities of the spot rate dynamics, moreover the spot rate is assumed to be Markovian and
consequently dr_t depends only on r_t. The fundamental theorem of asset pricing places no
restrictions on what the drift should be since the spot rate is not the price of an asset. The
advantage of this method is that one is able to price interest sensitive instruments without
having a look to the markets for these securities, therefore it would be helpful in determining
the price of the implicit option that by definition is not traded. Furthermore it takes the
advantage of the liquidity of the Italian GS market. However, the traditional approach has
several disadvantages. Because the drift of the instantaneous spot rate under the risk-neutral
martingale measure is not directly observable, it is almost impossible to verify the consistency
of the assumed dynamics with time series data. While it is possible to construct multi-factor
spot rate models in order to obtain a less than perfect correlation across spot (or forward) rates
of different maturities, it is difficult to calibrate exactly the correlations across the factors in
order to obtain a desired (empirical) correlation matrix across rates. No known short-rate
model is consistent with the Black formulas used by the market to quote prices for caps/floors
and swaptions and known spot-rate model can ensure an exact fit to a set of caps/floors or
swaption quotes. Yet, such an exact fit is very desirable for exotic derivatives traders who
hedge their trades using vanilla derivatives.

The Heath-Jarrow-Morton approach (HJM), on the other end, exploits the arbitrage relation
between forward rates and bond prices to impose restrictions on the dynamics of
instantaneous forward rates directly. By doing this it eliminates the need to model the
expected rate of change of r_t, still volatilities remain to be estimated. The HJM approach was
later applied to discrete-tenor forward rates in a series of papers that started with the work of
Brace-Gatarek-Musiela (1997). Models based on assumptions regarding the evolution of
discrete-tenor forward rates are known as BGM models, market models or LIBOR market models. The modern approach has several advantages over the classical approach. Because the volatilities of forward rates are the same under the martingale measure and under the true (historical) probability measures, the consistency of the assumed dynamics for forward rates with time series data can be easily verified. An exact fit of the initial term structure is obtained by construction, since the initial forward curve is exogenous to the model. Since assumptions are made directly on the volatilities of forward rates, it is easy to calibrate a model in order to obtain a desired (empirical) correlation matrix across forward rates. The "standard" market model leads to lognormally-distributed forward rates and is consistent with the Black formulas used to quote prices for caps/floors. In addition, it closely approximates the Black formulas used to quote prices for swaptions. Moreover, it is very easy to calibrate the "standard" market model to obtain an exact fit of a set of at the money cap/floor prices. One disadvantage of the modern pricing approach is that arbitrary specifications of the forward rate volatilities will in general lead to non-Markovian dynamics, thus requiring the simulation of a large number of state variables.

Brigo and Mercurio (2001) provide a deep analysis of these issues. The choice of the model may require also to set the number of risk factors and the eventual inclusion of a jump component in the interest rate process. However, before the model is chosen one has to consider also the market data that would be available to calibrate its parameters. Given the data available to me the model I chose is the one-factor time-homogeneous version of the Cox – Ingersoll –Ross model (CIR). The general equilibrium approach developed by Cox, Ingersoll and Ross (1985) led to the introduction of a square root term in the diffusion coefficient of the instantaneous short-rate dynamics proposed by Vasicek in 1977. The resulting model has been a benchmark for many years and it remains useful because of its analytical tractability and the fact that, contrary to the Vasicek (1977) one, the spot rate is always positive. Moreover, it provides closed form solutions for both the price of bonds and of plain vanilla options written on bonds. Under the risk neutral martingale measure the formulation of the model is:

\[ dr = \kappa(\theta - r)dt + \sigma \sqrt{r} dw \]

where \( \kappa \) is the speed of adjustment of the interest rate towards its long-run average \( \theta \), \( \sigma \sqrt{r} \) is the volatility of changes in the instantaneous interest rate and \( dw \) is a standardized Wiener
process. Moreover if $0 < \kappa < 1$ the process is mean-reverting, this means that the interest rate converges to its long-run value. It can be proved that the CIR model possesses an affine term structure, that is the price of a pure discount bond with residual maturity $\tau = T-t$ can be written as:

$$P(r,t,T) = F(t,T) e^{-G(t,T)r}$$

where

$$F(t,T) = \left( \frac{\phi_1 e^{\phi_\tau r}}{\phi_2 (e^{\phi_\tau r} - 1) + \phi_1} \right)^{\phi_3}$$

and

$$G(t,T) = \frac{e^{\phi_\tau r} - 1}{\phi_2 (e^{\phi_\tau r} - 1) + \phi_1}$$

Subject to the boundary condition

$$P(r,T,T)=1$$

This means that the price of a pure discount bond with residual maturity $\tau$ is a function of the state variable $r$ and of the three parameters $\phi_1$, $\phi_2$ and $\phi_3$ where:

$$\phi_1 = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2}$$

$$\phi_2 = \frac{\kappa + \lambda + \phi_1}{2}$$

$$\phi_3 = \frac{2\kappa \theta}{\sigma^2}$$

and $-\lambda$ is as usual the market price of risk. Given $P(r,t,T)$ the whole term structure of interest rates $R(r,t,T)$ can be worked out using the following relationship:
\[ R(r, t, T) = -\frac{\ln[P(r, t, T)]}{T - t}. \]

In particular, we have that:

\[ R_0 = \lim_{t \to 0} R(r, t, T) = r \]
\[ R_\infty = \lim_{T \to \infty} R(r, t, T) = (\phi_1 - \phi_2)\phi_3 \]

and

\[ \sigma = \sqrt{2(\phi_1\phi_2 - \phi_3^2)}. \]

If we substitute the value of \( R_\infty \) for \( \theta \) we can gain economic intuition on the three parameters. In fact, now they can be written as:

\[ \phi_1 = \kappa - \lambda \]
\[ \phi_2 = \kappa \]
\[ \phi_3 = -\frac{\theta}{\lambda}. \]

The CIR model is easy to implement and this property is very important for a financial institution. For this reason, the CIR model is often used for practical purposes. As our aim is to price an option that is part of the Specialists’ privileges and that they may want to price, then it is useful to start with instruments already used by banks and managers.

Of the Italian GS trading on a given date there are both coupon and zero coupon bonds, this must be taken into account when calibrating the CIR model. Moreover, the quoted prices are clean prices, meaning they do no incorporate the coupon that has been maturing from the last payment date. Hence we have to work out the accrued interest in order to obtain the dirty price or \textit{cum-coupon} price. The CIR model is estimated using the one-stage approach developed by Brown and Dybvig (1986). This assumes that if we consider a coupon bond that entitles the holder to a vector of remaining payments, \textit{cf}, to be received in a vector of dates, \textit{d}, the value of such bond at time \( t \) is equal to
\[ V^*(t, cf, d) = \sum_{d_i} cf_i P(r, t, d_i). \]

Hence a zero coupon bond can be represented as a bond with a single payment at its maturity date. To estimate the parameters of the model we make the assumption that the bond price \( V \) at time \( t \) deviate by the model price \( V^* \) by an error term, \( \varepsilon_t \):

\[ V(t, cf, d) = V^*(t, cf, d) + \varepsilon_t. \]

While Brown and Dybvig assume that the error term is zero-mean and independent and identically distributed as a Normal, I follow Barone, Cuoco and Zautieck (1989) and assume that is increasing in the bond’s duration. Indeed, it is natural to assume that a pricing error is smaller the closer is the bond’s maturity date. This implies that in order to make the error term homoskedastic both sides of the previous equations are divided by the square root of the product between the modified duration and the cum-coupon price of the bond. The resulting model is of the kind

\[ Y = X P(r, \beta) + u \]

where \( X \) is the cash flows’ matrix and \( \beta \) is the vector of the parameters.

**Section II**

The model is calibrated on daily data on Italian GS for the quarters 2004:1 – 2006:1 for a total of 579 trading days. These data come from MTS Time Series database. MTS Time Series package contains both high frequency tick data and daily data. In particular, I use daily trade and quote information taken at 5.00 p.m. Central European Time (CET) and an identifying cross-sectional file providing bond descriptions. For each day I have the closing mid price of all the bonds and notes traded that day, the time to maturity, the coupon payments, the day counting market conventions. Moreover the database includes the accrued interest and the modified duration. When the last two quantities were missing they have been worked out. Data cover a wide range of maturities, from one day up to February 1st, 2037 and are ordered by quote date and then by maturity date.
The CIR model has been estimated in many different ways [For estimations using non-linear regression techniques, see S. Brown and P. Dybvig (1986) and R. Brown and S. Schaefer (1988). For an estimation based on the method of moments, see M. Gibbons and K. Ramaswamy (1986). An alternative method, known as the two-stage method, consists in first estimating the parameters $\kappa$, $\mu$ and $\sigma$ of process followed by the instantaneous interest rate, using the time series of a short-term rate, and then $\lambda$ on the basis of equation (3): see A. Ananthanarayanan and E. Schwartz (1980). For an application of the two-stage method to the Italian market, see E. Barone and R. Cesari (1986)]. Here I chose to calibrate the model to the available data using a non-linear least squares method. This is a numerical method that find the minimum of a function in a recursive way, starting from a parameter vector arbitrary chosen. From a geometrical point of view, this method can be thought as a path that, from a starting point, chooses the way that brings to the nearest deep valley down. Anyway, it could be possible that the nearest deep valley is not the deepest valley, that is the non-linear least squares method could bring only to a local minimum down. So, the choice of the starting point is very delicate. In general we do not know, neither approximately, where the global minimum lies. This means that in general we can not choose a reasonable starting point for the parameter vector sufficiently close to the minimum. A rough but efficient method to overcome this problem is suggested by Torosantucci Uboldi (200?). They propose to build a net of starting points, calibrate the model using $n$ set of starting points from the grid, and then to chose the parameters that ensure both a reasonable expectation of the market, and the mathematical hypothesis of the CIR model. If more than one set of such parameters satisfies these requirements they chose the set that minimize their score function. The main drawback of this method is the long time necessary to give a solution. For this reason they suggest to restrict the number of iterations either by applying this procedure on few distant days, or to apply it just on the first day and then use that day’s local minimum as starting point of the following days calibration. The problem with this method is that if at day $t$ a strange and particular market situation occurs it is reasonable to obtain an anomalous minimum parameter vector.

Given that per each day I have about 60 observations, it should not be problematic to find the global minimum independently by the chosen starting point. I then used a procedure that is close to Torosantucci and Uboldi’s (henceforth TU), robust as well but less time consuming. First I fixed an interval for each element of the parameter’s vector. According to TU the intervals should be large enough to include the global minimum. In order to limit the
dimension of the intervals I chose them in such a way that they include the values found in
previous calibrations of the CIR model on Italian data (add reference). Then I made a draw of
the first starting point from the uniform distributions functions built on the chosen intervals,
and I calibrated the CIR model for all the days in the sample using the selected starting point.
I stored the result and then made a second draw and another calibration. At the end of the
second simulation I compared the resulting minima with the previous ones. Then I saved the
best ones coming from this comparison into a separate file, where for best selection criterion I
used, accordingly with the non-linear least square method, the sum of square errors. I repeated
this procedure 100 times. The result was a file containing 101 different values for each
parameter on each day and a file including the time series of the parameters minimizing the
function. This procedure allows each set of parameters to come from a different set of starting
points. In order to check for the stability of the obtained parameters, I also counted the
number of revisions on parameters coming from the 101 simulations. The resulting
parameters are summarized by year in the following table.

Table 2.1 Parameters’ statistics

<table>
<thead>
<tr>
<th>years</th>
<th># obs.</th>
<th>mean</th>
<th>sd</th>
<th>min</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>258</td>
<td>0.2186</td>
<td>0.0348</td>
<td>0.1326</td>
<td>0.3512</td>
</tr>
<tr>
<td>2005</td>
<td>257</td>
<td>0.1779</td>
<td>0.0287</td>
<td>0.0791</td>
<td>0.2500</td>
</tr>
<tr>
<td>2006</td>
<td>64</td>
<td>0.2386</td>
<td>0.0529</td>
<td>0.1664</td>
<td>0.4154</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>years</th>
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<thead>
<tr>
<th>years</th>
<th># obs.</th>
<th>mean</th>
<th>sd</th>
<th>min</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>258</td>
<td>4.24</td>
<td>1.63</td>
<td>3.23</td>
<td>13.87</td>
</tr>
<tr>
<td>2005</td>
<td>257</td>
<td>5.52</td>
<td>3.35</td>
<td>3.69</td>
<td>13.90</td>
</tr>
<tr>
<td>2006</td>
<td>64</td>
<td>5.66</td>
<td>3.46</td>
<td>3.36</td>
<td>13.78</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>years</th>
<th># obs.</th>
<th>mean</th>
<th>sd</th>
<th>min</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>258</td>
<td>0.0180</td>
<td>0.0016</td>
<td>0.0140</td>
<td>0.0254</td>
</tr>
<tr>
<td>2005</td>
<td>257</td>
<td>0.0195</td>
<td>0.0023</td>
<td>0.0161</td>
<td>0.0279</td>
</tr>
<tr>
<td>2006</td>
<td>64</td>
<td>0.0259</td>
<td>0.0013</td>
<td>0.0238</td>
<td>0.0283</td>
</tr>
</tbody>
</table>

I ended up with ??? starting points for the 579 days in the sample.
In Table 2.2 are reported statistics about the number of parameters’ revisions generated by the 101 simulations. On average 2005 and the beginning of 2006 exhibit a higher number of revisions.

**Table 2.2 Number of revisions**

<table>
<thead>
<tr>
<th>years</th>
<th># obs.</th>
<th>mean</th>
<th>sd</th>
<th>min</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>258</td>
<td>5.01</td>
<td>0.88</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>2005</td>
<td>257</td>
<td>5.96</td>
<td>0.84</td>
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<td>10</td>
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<tr>
<td>2006</td>
<td>64</td>
<td>6</td>
<td>1.49</td>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

Further information about parameters’ stability can be inferred from the following figures. In particular, Figure 2.1 shows the time series of the parameters. Although I did not exclude any day from the sample, the resulting parameters seem to be pretty stable. This holds in particular for $\phi_1$, $\phi_2$ and $r$. The parameter $\phi_3$ shows many peaks, some of them coincide with auction and reopenings days (Add analysis). This path is particularly unstable in the last part of the sample. Figure 2.2 shows the time series of the diffusion coefficient and of the long term interest rate respectively. The long term rate’s peak is probably due to liquidity problems since it happens on December 24th 2004. The last two figures shows the time series evolution of the volatility and of the sum of square errors generated by the model.
Figure 2.2
Given the calibrated parameters it is possible to price the option. Let $c(r,t,T,s,K)$ be the price of a call option with strike price $K$ and maturity $T$, written on a pure discount bond with
maturity \( s (s > T > t) \). The price of the option, obtained solving the problem (Insert PDE+boundary cond) is

\[
c(r,t,T,s,cf,K) = P(r,t,s)\chi^2(d_1,df_1,nc_1) - KP(r,t,T)\chi^2(d_2,df_2,nc_2)
\]

where \( \chi^2(d,df,nc) \) is the non-central chi-square distribution function valued at point \( d \), with \( df \) degree of freedom and non-centrality parameter \( nc \). These parameters are specified both by Cox, Ingersoll and Ross (1985) and Jamshidian (1990). In what follows I will refer to the CIR (1985) ones:

\[
d_1 = 2r^*[\varphi + \psi + G(T,s)]
\]

\[
d_2 = 2r^*(\varphi + \psi)
\]

\[
df_1 = df_2 = 2\phi
\]

\[
nc_1 = \frac{2\phi^2 re^{\varphi}(T-r)}{\varphi + \psi + G(T,s)}
\]

\[
nnc_2 = \frac{2\phi^2 re^{\varphi}(T-r)}{\varphi + \psi}
\]

\[
\varphi = \frac{2\phi_1}{\sigma^2 [e^{\phi(T-r)} - 1]}
\]

\[
\psi = \frac{2\phi_2}{\sigma^2}
\]

\[
r^* = \ln \left[ \frac{F(T,s)}{K G(T,s)} \right]
\]

Where \( r^* \) is the critical interest rate below which the option will be exercised, and it is obtained solving \( P(r^*,T,s) = K \) with respect to \( r^* \). The option formulation tells us that the replicating strategy is simply to buy \( \chi^2(d_1,df_1,nc_1) \) pure discounts bonds with maturity \( s \) and sell \( K\chi^2(d_2,df_2,nc_2) \) pure discount bonds with maturity \( T \). Since the options to be considered here are written on coupon bonds, we have to take it into account. In particular with one-factor model it can be proved that the option written on a coupon bond is equivalent to a portfolio of options on pure discount bonds (Jamshidian 1989, Longstaff 1990):
\[ c(r,t,T,s,cf,K) = \sum_{j=q}^{m} cf_j c(r,t,T,s_j,K_j,0). \]

Hence, given the formula for the price of an option on pure discount bonds \( c \), we obtain

\[ c(r,t,T,s,cf,K) = \sum_{j=q}^{m} cf_j P(r,t,s_j) \chi^2(d_{1,j}, df_{1,j}, nc_{1,j}) - K_j P(r,t,T) \chi^2(d_{2,j}, df_{2,j}, nc_{2,j}) \]

where \( cf_j \) is the bond’s payments at time \( s_j \) \((T < s_q \leq s_j \leq s_m)\), \( r^* \) is the solution of \( K = \sum_{j=q}^{m} cf_j P(r^*,t,s_j) \) and \( K_j \) the solution of \( K_j = P(r^*,t,s_j) \).

In order to work out the value of the option I use the parameters calibrated the closing prices of the day before the ordinary auction.\(^5\) The results of the option pricing for the reopenings occurred during the year 2004 are reported in the following table.

**Table 2.3 Option Pricing Results**

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\(^5\) These values are the most recent available before the auction.
The first and the second columns contain information about the date of the ordinary auctions and the characteristic of the bonds auctioned, namely if it is a BTP or a 2-year CTZ, the maturity date and the coupon rate. The price of the call option written on the described security is in the third column. The fourth column contains the option’s strike price, the remaining report the parameters resulting from the calibration of the CIR model on the closing price of the day before the ordinary auction takes place.
References


Brandolini, A. 2004,”Valutazione Finanziaria della Riapertura delle Aste dei Titoli di Stato a Dieci Anni”, Università di Roma Tor Vergata, mimeo.


Decreto 13 maggio 1999, n. 219, "Disciplina dei mercati all’ingrosso dei titoli di Stato".