Minireview

The application of continuous-time random walks in finance and economics

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Abstract

This paper reviews some applications of continuous time random walks (CTRWs) to Finance and Economics. It is divided into two parts. The first part deals with the connection between CTRWs and anomalous diffusion. In particular, a simplified version of the well-scaled transition of CTRWs to the diffusive or hydrodynamic limit is presented. In the second part, applications of CTRWs to the ruin theory of insurance companies, to growth and inequality processes and to the dynamics of prices in financial markets are outlined and briefly discussed.

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1. Introduction

In 1965, Montroll and Weiss published a paper on continuous-time random walks (CTRWs) [1], in which the waiting-time between two consecutive jumps of a diffusing particle is a real positive stochastic variable. It was the starting point for several developments on the physics of normal and anomalous self-diffusion.

Anomalous relaxation with power-law tails of the waiting-time density was investigated by Montroll and Scher who used Monte Carlo simulations [2]. Many papers were devoted by Shlesinger, Tunaley and other authors to the long-time asymptotic behaviour of CTRWs [3–7] (see also Ref. [8]). The relevance of the Mittag-Leffler function in anomalous relaxation was discussed by Caputo and Mainardi [9].

Further developments on the theory of CTRWs, including the problem of anomalous relaxation, can be found in the following references [10–23]. A book by ben-Avraham and Havlin discusses the applications of the formalism presented in the aforementioned papers to real physical systems [24].

The asymptotic relation between properly scaled CTRWs with power-law distributed waiting times and fractional diffusion processes was rigorously analysed by Balakrishnan already in 1985, dealing with anomalous diffusion in one dimension [25], 4 years before the publication of the fundamental paper by Schneider and Wyss on the analytic theory of fractional diffusion and wave equations [26]. Later, many other authors studied this subject from slightly different viewpoints [27–37]. The correspondence between CTRWs
with Mittag-Leffler distributed waiting times and the time-fractional diffusion equation has been lucidly worked out and explained in Ref. [30] by Hilfer and Anton (see also Refs. [38,39]). They have shed light on the relevance of the Mittag-Leffler function, but their specific aims, methods and interpretations are completely different from those of Balakrishnan. However, it must be recognized that already Balakrishnan has found, as the natural choice for the waiting-time in CTRWs approximating fractional diffusion, the waiting-time density whose Laplace transform is (in the notation used in this paper) \(1/(1 + cs^\beta)\), where \(c\) is a positive constant. Implicitly, this is the Mittag-Leffler waiting-time distribution presented in Section 2 of this paper.

This intense theoretical activity has been thoroughly reviewed by Metzler and Klafter [17,40] and by Piryatinska et al. [41] and is well known among physicists working on anomalous diffusion. It is less popular in the interdisciplinary field of Econophysics. The application of CTRWs to various problems in Finance and Economics is even less known. Already in 1903, the pioneering thesis of Philip Lundberg led to the Cramér–Lundberg theory of insurance failure, where claims to be paid by an insurance company are modelled via a compound Poisson process, an instance of CTRW [42,43]. Moreover, mathematicians and probabilists have given several important and interesting contributions to the theory of CTRWs [44–50]. Unfortunately, the various communities that study and use CTRWs ignore each other and results in one field are not always taken into account in other fields.

The present review paper is intended as a first attempt to fill this gap, at least for physicists, and give some pointers to the relevant literature. It is divided as follows. Section 2 is devoted to the theory of CTRWs and their connection with the so-called fractional diffusion equation. This material has already been presented in this form in Ref. [51], where only applications to finance are discussed. Section 3 describes applications of CTRWs to the theory of insurance failure, to the theory of growth and inequality in Economics and to high-frequency price dynamics in financial markets. Finally, in Section 4, conclusions are drawn and directions for future research outlined.

It must be remarked that the applications of CTRWs to Finance and Economics listed above and studied below reflect the interest of the author and are not the only possibilities. Various applications are possible in the field of queueing theory and it is worth mentioning the work of Hilfer on sales forecast and planning [52].

It is the hope of the present author to encourage further research on the applications of CTRWs to interdisciplinary problems. As the reader will see, they are a very useful and handy theoretical tool and it is possible to generalize them to more complex processes that can always be studied by means of Monte Carlo simulations, if not analytically. However, this review paper also reflects knowledge limits and is still far from a complete coverage of the subject.

2. Theory

CTRWs are point processes with reward [45], a special instance of pure jump processes. The point process is characterized by a sequence of independent identically distributed (i.i.d.) positive random variables \(\tau_i\), which can be interpreted as waiting times between two consecutive events:

\[
t_n = t_0 + \sum_{i=1}^{n} \tau_i; \quad t_n - t_{n-1} = \tau_n; \quad n = 1, 2, 3, \ldots; \quad t_0 = 0.
\]  

Pure jump processes are Markovian if and only if the waiting times follow the exponential distribution [49]. The rewards are i.i.d. random variables: \(\xi_i\). In the usual physical interpretation, the \(\xi_i, s\) represent the instantaneous jumps of a diffusing particle, and they can be \(n\)-dimensional vectors. If diffusion takes place on an \(n\)-dimensional lattice, these jumps occur between discrete positions \(x_j\), where \(j\) is an integer index. In this paper, only the one-dimensional case is studied, and both jumps and positions are considered as continuous variables, but the extension of many results to the \(n\)-dimensional case and to lattices is straightforward. The position \(x\) of the particle at time \(t\) is given by the following sum (with the random variable \(n(t)\) defined by \(n(t) = \max\{m : t_m \leq t\}\) and \(x(0) = 0\)):

\[
x(t) = \sum_{i=1}^{n(t)} \xi_i.
\]

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In the next section, the rewards \( \xi \)'s as well as the positions \( x(t) \) will assume different meanings according to the specific problem addressed. The time series \( \{x(t_i)\} \) can be characterized by \( \phi(\xi, \tau) \), the joint probability density of jumps \( \xi = x(t_{i+1}) - x(t_i) \) and of waiting times \( \tau = t_{i+1} - t_i \). The joint density satisfies the normalization condition \( \int \int d\xi d\tau \phi(\xi, \tau) = 1 \). It must be again remarked that both \( \xi \) and \( \tau \) are assumed to be independent and identically distributed (i.i.d.) random variables. This strong assumption is useful to derive limit theorems for the stochastic processes described by CTRWs. However, in various applications, the presence of heteroskedasticity, as well as correlations between waiting times do falsify the i.i.d hypothesis. This is the case in finance. The reader interested in a review on correlated random variables is referred to Chapter 8 in McCauley’s recent book [53].

In general, jumps and waiting times are not independent from each other [54]. Probabilistic arguments (see Refs. [1,37,55]) can be used to derive the following integral equation that gives the probability density, \( p(x, t) \), for the particle of being in position \( x \) at time \( t \), conditioned by the fact that it was in position \( x = 0 \) at time \( t = 0 \):

\[
p(x, t) = \delta(x) \Psi(t) + \int_0^t \int_{-\infty}^{+\infty} \phi(x - x', t - t') p(x', t') \, dt' \, dx',
\]

where \( \delta(x) \) is Dirac's delta function and \( \Psi(t) \) is the so-called survival function. \( \Psi(t) \) is related to the marginal waiting-time probability density \( \psi(\tau) \). The two marginal densities \( \psi(\tau) \) and \( \lambda(\xi) \) are:

\[
\psi(\tau) = \int_{-\infty}^{+\infty} \phi(\xi, \tau) \, d\xi,
\]

\[
\lambda(\xi) = \int_0^{+\infty} \phi(\xi, \tau) \, d\tau
\]

and the survival function \( \Psi(t) \) is

\[
\Psi(t) = 1 - \int_0^t \psi(\tau') \, d\tau' = \int_t^{+\infty} \psi(\tau') \, d\tau'.
\]

Eq. (3) is generally valid for pure jump processes and its derivation can be found also in introductory textbooks on stochastic processes [49]. It can be obtained by recalling the meaning of the survival function: the probability of no jumps taking place up to time \( t \). The first term in the sum on the right-hand side takes into account the permanence in \( x = 0 \) up to time \( t \), whereas the second term gives the probability density of jumping from the position \( x' \) at time \( t' \) to the position \( x \) at time \( t \). If waiting times are exponentially distributed, CTRWs are also known as compound Poisson processes. In this case, CTRWs are Markovian processes that belong to the class of Lévy processes, that is processes with stationary and independent increments with the distribution characterized by the one point probability density \( p(x, t) \) being infinitely divisible. Conversely, for every infinite divisible distribution, there exists a corresponding Lévy process [56].

The integral equation (3), can be solved in the Laplace–Fourier domain. The Laplace transform, \( \tilde{g}(s) \) of a (generalized) function \( g(t) \) is defined as

\[
\tilde{g}(s) = \int_0^{+\infty} dt \, e^{-st} g(t),
\]

whereas the Fourier transform of a (generalized) function \( f(x) \) is defined as

\[
\hat{f}(\kappa) = \int_{-\infty}^{+\infty} dx \, e^{i\kappa x} f(x).
\]

A generalized function is a distribution (like Dirac’s \( \delta \)) in the sense of Sobolev and Schwartz [57]. One gets:

\[
\tilde{p}(\kappa, s) = \tilde{\Psi}(s) \frac{1}{1 - \tilde{\psi}(\kappa, s)},
\]
or, in terms of the density $\psi(\tau)$:

$$\tilde{p}(\kappa, s) = \frac{1 - \tilde{\psi}(s)}{s} \frac{1}{1 - \tilde{\phi}(\kappa, s)},$$

(9)

as, from Eq. (5), one has

$$\Psi(s) = \frac{1 - \tilde{\psi}(s)}{s}.$$  

(10)

In order to obtain $p(x, t)$, it is then necessary to invert its Laplace–Fourier transform $\tilde{p}(\kappa, s)$. For jumps independent from waiting times, a series solution to the integral equation (3) exists. This will be derived in the next subsection.

2.1. Limit theorems: the uncoupled case

In the so-called uncoupled case, the joint probability density of jumps and waiting times can be written as the product of the two marginal densities:

$$\varphi(\xi, \tau) = \lambda(\xi)\psi(\tau),$$

(11)

with the normalization conditions $\int d\xi \lambda(\xi) = 1$ and $\int d\tau \psi(\tau) = 1$. This case has been discussed by Gorenflo, Mainardi and the present author [37]. In the following, the main results of that paper are summarized and presented in a simplified version that uses the method of the characteristic function to obtain the diffusive limit of the CTRW.

In the uncoupled case, the integral master equation for $p(x, t)$ becomes:

$$p(x, t) = \delta(x)\Psi(t) + \int_0^t \psi(t - \tau) \left[ \int_{-\infty}^{+\infty} \lambda(x - \xi)p(x', \tau) d\xi \right] d\tau.$$

(12)

This equation has an explicit solution in terms of $P(n, t)$, the probability of $n$ jumps occurring up to time $t$, and of the $n$-fold convolution of the jump density, $\lambda_n(x)$:

$$\lambda_n(x) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} d\xi_{n-1} \ldots d\xi_1 \lambda(x - \xi_{n-1}) \ldots \lambda(\xi_1).$$

(13)

Indeed, $P(n, t)$ is given by

$$P(n, t) = \int_0^t \psi_n(t - \tau)\Psi(\tau) d\tau,$$

(14)

where $\psi_n(\tau)$ is the $n$-fold convolution of the waiting-time density:

$$\psi_n(\tau) = \int_0^\tau \ldots \int_0^{\tau_1} d\tau_{n-1} \ldots d\tau_1 \psi(\tau - \tau_{n-1}) \ldots \psi(\tau_1).$$

(15)

The $n$-fold convolutions defined above are probability density functions for the sum of $n$ independent variables. Eqs. (13) and (15) can be used thanks to the i.i.d. hypothesis.

Using the Laplace–Fourier method and recalling the properties of Laplace–Fourier transforms of convolutions, one gets the following solution of the integral equation [37,58–60]:

$$p(x, t) = \sum_{n=0}^{\infty} P(n, t)\lambda_n(x).$$

(16)

Eq. (16) can be directly derived by pure probabilistic considerations from Eq. (2) and used as the starting point to derive Eq. (12) via the transforms of Fourier and Laplace, as it describes a jump process subordinated to a renewal process [61–63].
It is now interesting to consider the following pseudodifferential equation, giving rise to anomalous relaxation and power-law tails in the waiting-time probability [9]:

$$\frac{d^\beta}{dt^\beta} \Psi(\tau) = -\Psi(\tau), \quad \tau > 0, \quad 0 < \beta \leq 1; \quad \Psi(0^+) = 1,$$

(17)

where the operator $d^\beta/dt^\beta$ is the Caputo fractional derivative, related to the Riemann–Liouville fractional derivative. For a sufficiently well-behaved function $f(t)$, the Caputo derivative [64] is defined by the following equation, for $0 < \beta < 1$:

$$\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t f(\tau) \frac{d\tau}{(t-\tau)^\beta} - \frac{t^{-\beta}}{\Gamma(1-\beta)} f(0^+)$$

(18)

and reduces to the ordinary first derivative if $\beta = 1$. The Laplace transform of the Caputo derivative of a function $f(t)$ is

$$L \left( \frac{d^\beta}{dt^\beta} f(t); s \right) = s^\beta \tilde{f}(s) - s^{\beta-1} f(0^+).$$

(19)

If Eq. (19) is applied to the Cauchy problem of Eq. (17), one gets:

$$\tilde{\Psi}(s) = \frac{s^{\beta-1}}{1 + s^\beta}.$$

(20)

Eq. (20) can be inverted, giving the solution of Eq. (17) in terms of the Mittag-Leffler function of parameter $\beta$ [9,65,66]:

$$\Psi(\tau) = E_\beta(-\tau^\beta),$$

(21)

defined by the following power series in the complex plane:

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}.$$

(22)

The Mittag-Leffler function is a possible model for a fat-tailed survival function. For $\beta = 1$, the Mittag-Leffler function coincides with the ordinary exponential function. In the other cases, for small $\tau$, the Mittag-Leffler survival function coincides with the stretched exponential:

$$\Psi(\tau) = E_\beta(-\tau^\beta) \simeq 1 - \frac{\tau^\beta}{\Gamma(\beta + 1)} \simeq \exp(-\tau^\beta/\Gamma(\beta + 1)), \quad 0 \leq \tau \ll 1,$$

(23)

whereas for large $\tau$, it has a power-law asymptotic representation:

$$\Psi(\tau) \sim \frac{\sin(\beta \pi)}{\pi} \frac{\Gamma(\beta)}{\tau^\beta}, \quad 0 < \beta < 1, \quad \tau \to \infty.$$

(24)

Accordingly, for small $\tau$, the probability density function of waiting times $\psi(\tau) = -d\Psi(\tau)/d\tau$ behaves as

$$\psi(\tau) = -\frac{d}{d\tau} E_\beta(-\tau^\beta) \simeq \frac{\tau^{-(1-\beta)}}{\Gamma(\beta)}, \quad 0 \leq \tau \ll 1$$

(25)

and the asymptotic representation is

$$\psi(\tau) \sim \frac{\sin(\beta \pi)}{\pi} \frac{\Gamma(\beta + 1)}{\tau^{\beta+1}}, \quad 0 < \beta < 1, \quad \tau \to \infty.$$

(26)

The Mittag-Leffler function is important as, without passage to the diffusion limit, it leads to a time-fractional master equation, just by insertion into the CTRW integral equation. This fact was discovered and made explicit for the first time in 1995 by Hilfer and Anton [38], who gave a rigorous proof of equivalence between CTRWs and fractional diffusion. Therefore, this special type of waiting-time law (with its particular properties of being singular at zero, completely monotonic and long-tailed) may be best suited for approximate CTRW Monte Carlo simulations of fractional diffusion.
For processes with survival function given by the Mittag-Leffler function, the solution of the master equation can be explicitly written [37] as

\[ p(x, t) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} E_\beta^n (-t^\beta) \lambda_\beta(x), \] (27)

where

\[ E_\beta^n (z) = \frac{d^n}{dz^n} E_\beta(z). \]

The Fourier transform of Eq. (27) is the characteristic function of \( p(x, t) \) and is given by

\[
\hat{p}(\kappa, t) = E_\beta \left[ \frac{t^\beta}{r^\beta} (\hat{\lambda}(h) - 1) \right].
\] (28)

If log-returns and waiting times are scaled according to

\[ x_n(h) = h \xi_1 + h \xi_2 + \cdots + h \xi_n \] (29)

and

\[ t_n(r) = r \tau_1 + r \tau_2 + \cdots + r \tau_n, \] (30)

the scaled characteristic function becomes:

\[
\hat{p}_{h,r}(\kappa, t) = E_\beta \left[ \frac{t^\beta}{r^\beta} (\hat{\lambda}(h \kappa) - 1) \right].
\] (31)

Now, if we assume the following asymptotic behaviours for vanishing \( h \) and \( r \):

\[
\hat{\lambda}(h \kappa) \sim 1 - h^2 |\kappa|^2, \quad 0 < \alpha \leq 2
\] (32)

and

\[
\lim_{h,r \to 0} \frac{h^\alpha}{r^\beta} = 1,
\] (33)

we get that

\[
\lim_{h,r \to 0} \hat{p}_{h,r}(\kappa, t) = \hat{u}(\kappa, t) = E_\beta [-t^\beta |\kappa|^2].
\] (34)

The Laplace transform of Eq. (34) is

\[
\hat{u}(\kappa, s) = \frac{s^{\beta-1}}{|\kappa|^2 + s^\beta}.
\] (35)

Therefore, the well-scaled limit of the CTRW characteristic function coincides with the Green function of the following pseudodifferential \( \text{fractional diffusion equation} \):

\[
|\kappa|^2 \hat{u}(\kappa, s) + s^{\beta-1} \hat{u}(\kappa, s) = s^\beta - 1,
\] (36)

with \( u(x, t) \) given by

\[
u(x, t) = \frac{1}{\rho^{\beta/2}} W_{x, \beta} \left( \frac{x}{\rho^{\beta/2}} \right),
\] (37)

where \( W_{x, \beta}(u) \) is

\[
W_{x, \beta}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{-iku} E_\beta (-|k|^2),
\] (38)

the inverse Fourier transform of a Mittag-Leffler function [17,36,37,40,67,68].

For \( \beta = 1 \) and \( \alpha = 2 \), the fractional diffusion equation becomes the ordinary diffusion equation and the function \( W_{2,1}(u) \) is nothing else but the Gaussian probability density function evolving in time with a variance \( \sigma^2 = 2t \). In the general case (0 < \( \beta < 1 \) and \( 0 < \alpha < 2 \)), the function \( W_{x, \beta}(u) \) is still a probability density evolving...
The scaling equation, Eq. (33), can be written in the following form:

\[ h \sim r^{\beta/z}. \]  

(39)

If \( \beta = 1 \) and \( z = 2 \), one can recognize the scaling relation typical of Brownian motion (or the Wiener process).

In the passage to the limit outlined above, \( \bar{p}_{\tau,h}(\kappa,s) \) and \( \bar{u}(\kappa,s) \) are asymptotically equivalent in the Laplace–Fourier domain. Then, the asymptotic equivalence in the space–time domain between the master equation and the fractional diffusion equation is due to the continuity theorem for sequences of characteristic functions, after the application of the analogous theorem to sequences of Laplace transforms [48]. Therefore, there is convergence in law or weak convergence for the corresponding probability distributions and densities. Here, weak convergence means that the Laplace transform and/or Fourier transform (characteristic function) of the probability density function are pointwise convergent (see Ref. [48]).

In Eq. (28), let \( \beta = 1 \). This case corresponds to exponentially distributed waiting times and one has:

\[ \hat{p}(\kappa,t) = \exp[t(\hat{\lambda}(\kappa) - 1)] = [\exp(\hat{\lambda}(\kappa) - 1)]' = [\hat{p}(1)]', \]  

(40)

which is a property valid for characteristic functions of Lévy processes. In the case \( \beta = 1 \), the limiting probability density (Eq. (37)) becomes:

\[ u(x,t) = \frac{1}{t^{1/z}} W_{z,1} \left( \frac{x}{t^{1/2}} \right), \]  

(41)

where

\[ W_{z,1}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\kappa e^{-iu\kappa} \exp(-|\kappa|^z). \]  

(42)

The distribution defined by the probability density \( u(x,t) \) in Eq. (41) turns out to be infinite divisible and, for \( 0 < z < 2 \), there are corresponding Lévy processes that are called \( z \)-stable Lévy flights [69]. For \( z = 2 \) the corresponding Lévy process is the Bachelier–Wiener process.

For \( 0 < \beta < 1 \), Eq. (28) cannot be written in the form of Eq. (40). The generalized compound Poisson processes are not Lévy processes and they are non-Markovian, and this is true also in the diffusive limit.

### 2.2. Limit theorems: the coupled case

The diffusive limit in the coupled case is well discussed by Meerschaert et al. [34] based on the findings of Meerschaert and Scheffler [70]. The coupled case is relevant as, in general, jumps and waiting times are not independent [54,71]. Here, a condition that suffices to obtain the limiting pseudodifferential equation (36) will be given. This micro-theorem appeared for the first time in heuristic form in Ref. [72] and the rigorous proof was submitted to the proceedings of FDA 04 [73]. It is based on the results summarized in Ref. [37] and discussed in Refs. [74,75].

**Theorem.** Let \( \varphi(\xi,\tau) \) be the (coupled) joint probability density of a CTRW. If, under the scaling \( \xi \to h\xi \) and \( \tau \to rt \), the Fourier–Laplace transform of \( \varphi(\xi,\tau) \) behaves as follows:

\[ \tilde{\varphi}_{h,r}(\kappa,s) = \tilde{\varphi}(h\kappa,rs) \]  

(43)

and if, for \( h \to 0 \) and \( r \to 0 \), the asymptotic relation holds:

\[ \tilde{\varphi}_{h,r}(\kappa,s) = \tilde{\varphi}(h\kappa,rs) \sim 1 - \mu|h\kappa|^2 - v(rs)^\beta, \]  

(44)

with \( 0 < z \leq 2 \) and \( 0 < \beta \leq 1 \), then, under the scaling relation \( ph^z = w^\beta \), the solution of the (scaled) coupled CTRW master (integral) equation, Eq. (3), \( p_{h,r}(x,t) \), weakly converges to the Green function of the fractional diffusion equation, \( u(x,t) \), for \( h \to 0 \) and \( r \to 0 \).
Proof. The Fourier–Laplace transform of the scaled conditional probability density \( p_{h,r}(x,t) \) is given by
\[
\tilde{p}_{h,r}(\kappa,s) = \frac{1 - \tilde{\psi}(s)}{s} \frac{1}{1 - \tilde{\phi}(h\kappa,rs)}.
\] (45)

Replacing Eq. (44) in Eq. (45) and observing that \( \tilde{\psi}(s) = \tilde{\phi}(0,s) \), one asymptotically gets for small \( h \) and \( r \):
\[
\tilde{p}_{h,r}(\kappa,s) \sim \frac{\nu \rho^2 s^\beta - 1}{\nu \rho^2 s^\beta + \mu h^2 |\kappa|^2},
\] (46)

which for vanishing \( h \) and \( r \), under the hypotheses of the theorem, converges to
\[
\tilde{p}_{0,0}(\kappa,s) = \tilde{u}(\kappa,s) = \frac{s^\beta - 1}{s^\beta + |\kappa|^2},
\] (47)

where \( \tilde{u}(\kappa,s) \) is the Fourier–Laplace transform of the Green function of the fractional diffusion equation (see Eq. (36)). The asymptotic equivalence in the space–time domain, between \( p_{0,0}(x,t) \) and \( u(x,t) \), the inverse Fourier–Laplace transform of \( \tilde{u}(\kappa,s) \), is again ensured by the continuity theorem for sequences of characteristic functions, after the application of the analogous theorem for sequences of Laplace transforms [48]. There is convergence in law or weak convergence for the corresponding probability distributions and densities. \( \square \)

An important consequence of the above theorem is the following corollary showing that, in the case of marginal densities with finite first moment of waiting times and finite second moment of jumps, the limiting density \( u(x,t) \) is the solution of the ordinary diffusion equation (and thus the limiting process is the Wiener process).

**Corollary.** If the Fourier–Laplace transform of \( \phi(\xi,\tau) \) is regular for \( \kappa = 0 \) and \( s = 0 \), and, moreover, the marginal waiting-time density, \( \psi(\tau) \), has finite first moment, \( \tau_0 \), and the marginal jump density, \( \lambda(\xi) \), is symmetric with finite second moment, \( \sigma^2 \), then the limiting solution of the master (integral) equation for the coupled CTRW is the Green function of the ordinary diffusion equation.

**Proof.** Due to the hypothesis of regularity in the origin and to the properties of Fourier and Laplace transforms, we have that:
\[
\tilde{\phi}_{h,r}(\kappa,s) = \tilde{\phi}(h\kappa,rs) \sim \tilde{\phi}(0,0) + \frac{1}{2} \left( \frac{\partial^2 \tilde{\phi}}{\partial \kappa^2} \right)_{(0,0)} h^2 \kappa^2 + \left( \frac{\partial \tilde{\phi}}{\partial s} \right)_{(0,0)} rs
\]
\[
= 1 - \frac{\sigma^2}{2} h^2 \kappa^2 - \tau_0 rs
\] (48)

and as a consequence of the theorem, under the scaling \( h^2 \sigma^2/2 = \tau_0 r \), one gets, for vanishing \( h \) and \( r \):
\[
\tilde{p}_{0,0}(\kappa,s) = \tilde{u}(\kappa,s) = \frac{1}{s + k^2},
\] (49)

corresponding to the Green function (36) for \( \alpha = 2 \) and \( \beta = 1 \), that is the solution of the Cauchy problem for the ordinary diffusion equation. \( \square \)

3. Applications

In this section, three applications of CTRWs outside the realm of Physics will be presented. In all the three cases, there will be a discussion of the pioneering work that introduced the CTRW model and a short description of some recent developments related to the theory outlined in the previous section. Each of these applications would deserve a larger review, but pointers to relevant books and papers, if available, will be provided. The three applications are listed in chronological order of appearance, to the best of the knowledge of the author. The first one is the theory of ruin probability in insurance, introduced by Lundberg in 1903 [42]. The second application is based on Gibrat’s research on inequality in Economics and can be traced to Gibrat’s
monograph published in 1931 [76]. Finally, the first application of CTRWs to price fluctuations appeared in a paper written by Press in 1967 [77].

3.1. Theory of ruin probability in insurance

The theory of ruin probability for insurance companies is based on the following equation for the risk process, that gives the capital $R$ of such companies at time $t$ [42,43,78,79]:

$$R(t) = u + ct - \sum_{i=1}^{n(t)} \xi_i,$$

where $u$ is the initial capital, $c$ is the flux of capital due to the premium payment, and $\xi_i$ is a non-negative random variable representing the claims paid up to time $t$. Thus, the process of claim arrival is a CTRW where the jumps can only be zero or in the positive direction.

Ruin occurs when the capital $R(t)$ becomes zero. For a given claim process, the time to ruin depends on the initial capital $u$ and is defined as

$$\tau(u) = \inf\{t \geq 0 : R(t) < 0\}.\quad (51)$$

In this theory, other important quantities are the ruin probability in infinite time, $q(u)$, that is the probability that $\tau(u)$ is a finite quantity:

$$q(u) = \Pr[\tau(u) < \infty]\quad (52)$$

and, given a finite time horizon, $T$, the ruin probability in finite time, $q(u, T)$:

$$q(u, T) = \Pr[\tau(u) < T].\quad (53)$$

The ruin probability in finite time can be always estimated by means of Monte Carlo simulations, whereas the ruin probability in infinite time is only accessible through theoretical considerations [80].

As the analytical expressions, if available, of the ruin probability are rather cumbersome [79,81,82], some authors proposed to approximate the claim process by Brownian motion [83,84]. Recently, Furrer and co-workers applied this idea by means of the weak convergence theory outlined in the previous section in order to approximate the claim process by Brownian motion and $\alpha$-stable Lévy motion [85,86].

They consider claims whose distribution belongs to the basin of attraction of a Lévy $\alpha$-stable density $L_{\alpha}(u) = W_{\alpha,1}(u)$. In this hypothesis, they are able to derive the following approximation of the risk process:

$$R_{\alpha}(t) = u + ct - \lambda^{1/\alpha} D_{\alpha}(t),\quad (54)$$

where $D_{\alpha}(t)$ is a Lévy flight with unitary scale parameter, and $u, c, \lambda$ are positive constants.

Furrer has shown that the infinite time ruin probability for the previous approximated risk process is [87]:

$$q(u) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n! (1 + (\alpha - 1)m)^{u(\alpha - 1)m}},\quad (55)$$

where $a = c\lambda^{-1} \cos[\pi(\alpha - 2)/2]$.

3.2. Growth and inequality

The title of the book written by Gibrat is already a sort of manifesto [76]: Economic inequalities, application: to inequalities in wealth, in the concentration of enterprises, in the population of towns, in the statistics of families, etc., on a new law, the law of proportional effect. Gibrat claims the discovery of a simple law that can explain the origin of inequalities within Economic growth processes. Similar themes pervade much of the recent Econophysics literature [88,89].

A very simple non-conservative model that takes into account the ideas of Gibrat is the following:

$$s(n + 1) = \eta(n) s(n),\quad (56)$$
where \( s_i(n) \) represent the size of agent \( i \) and \( \eta(n) \) is a random variable always extracted from the same probability distribution at any discrete period \( n \).

Eq. (56) is a growth model with multiplicative noise. It is the basis on which the so-called generalized Lotka–Volterra models are built [90]. In this model, there are no direct or indirect interactions between the agents and the logarithm of the size, \( x(n) = \log[s(n)] \), is the sum of independent and identically distributed random variables, the growth rates, \( \xi = \log(\eta) \):

\[
x(n + 1) = x(0) + \sum_{m=1}^{n} \xi(m).
\]

If the growth rate is independent from the size, the Central Limit Theorem and its generalizations apply in the large \( n \) limit [91] and one gets either normal or Lévy distributed log-sizes and, therefore, log-normal or log-Lévy distributed sizes.

In a situation in which the growth shocks \( \xi \) can arrive at random times instead of at fixed periods, Eq. (57) must be replaced by its continuous-time version, given by the non-homogeneous case of Eq. (2):

\[
x(t) = x(0) + \sum_{m=1}^{n(t)} \xi_m,
\]

where \( n(t) = \max\{m : t_m \leq t\} \).

These ideas have been recently applied by a group of Italian economists to models of firm growth [92–95]. It is important to remark that, as shown in the previous section, \( p(x, t) \) follows either a normal or anomalous diffusive behaviour. Therefore, in the description of growth and inequality processes by means of Eq. (58), there is no equilibrium asymptotic probability density. On the contrary, conservative inequality processes also known as pure money exchange processes, such as those introduced by Angle and Bennati [96,97], do yield equilibrium distributions. Such conservative models are not suitable to describe economic growth, however.

### 3.3. High-frequency price fluctuation in financial markets

The mapping between CTRWs and high-frequency price fluctuations in financial markets was proposed in Ref. [72] along the lines presented in the papers by Clark and Parkinson [62,98]. Price dynamics is described in a pure phenomenological way. No assumption on the rationality or the behaviour of trading agents is necessary and also the efficient market hypothesis [99,100] can be neglected. Yet, the celebrated geometric Brownian motion model for financial price fluctuations [101] is a straightforward consequence of the mapping if some restrictive assumptions are considered [51,102].

Let \( S(t) \) denote the price of an asset at time \( t \). In a real market with a double-auction mechanism, prices are fixed when buy orders are matched with sell orders and a transaction (trade) occurs. Waiting times between two trades are a random variable. It is more convenient to refer to returns rather than prices. For this reason, the variable \( x(t) = \log S(t) \) is studied: the logarithm of the price or log-price. For a small price variation \( \Delta S = S(t_{i+1}) - S(t_i) \), the return \( r = \Delta S / S(t_i) \) and the logarithmic return \( r_{\log} = \log[S(t_{i+1})/S(t_i)] \) virtually coincide. In the financial interpretation, the jumps \( \xi_i \)'s have the meaning of log-returns, whereas the rewards \( x(t) \) represent log-prices at time \( t \).

The revisited and augmented CTRW formalism has been applied to high-frequency price dynamics in financial markets by our research group since 2000, in a series of three papers [72,55,103]. Later, the present author discovered a paper written by Press and published in 1967, where an instance of CTRWs was used: the so-called normal compound Poisson processes [77]. Other scholars have recently investigated this formalism [104–107], and new extensive data analyses based on the formalism have been performed [108–110].

The normal compound Poisson process has an exponential marginal waiting-time density \( \psi \) with average \( \tau_0 \) and a normal jump density, \( \lambda \) with variance \( \sigma^2 \) and average \( \theta \). Therefore, taking into account that the \( n \)-fold convolution of a normal density is still normal, with variance \( n\sigma^2 \) and average \( n\theta \), and that for \( \psi(t) = \exp(-t/\tau_0)/\tau_0 \) one has \( P(n, t) = \exp(-t/\tau_0)(t/\tau_0)^n/n! \); Eq. (16) yields for \( p(x, t) \):

\[
p(x, t) = \exp(-t/\tau_0) \sum_{n=0}^{\infty} \frac{(t/\tau_0)^n}{n!} \frac{1}{\sqrt{2\pi n\sigma}} \exp[-(x - n\theta)^2/2n\sigma^2].
\]
It is important to remark that, if the waiting times are exponentially distributed, then the number of jumps up to time $t$ follows the Poisson distribution and vice versa. The normal compound Poisson process is Markovian and, in the diffusive limit, its density is the Green function of the normal diffusion equation. Therefore, the limiting density for the log-price is normal and for the price is log-normal. Indeed, the geometric Brownian motion model is the limiting process for any compound Poisson process where rewards have a finite second moment. A third remark is the following. If one considers the distribution of log-returns sampled at fixed time intervals, $\Delta x = x(t + \Delta t) - x(t)$, because of the stationarity and homogeneity of the process, one has that $p(\Delta x) = p(x, \Delta t)$. It is possible to show that, in the case of a symmetric density ($\theta = 0$), one has for the kurtosis of $\Delta x$:

$$\beta_2 = 3 \frac{\tau_0}{\Delta t}. \quad (60)$$

Therefore, if a normal compound Poisson process is sampled at fixed times, with a sampling interval $\Delta t$, the distribution of log-returns, $\Delta x$, can be either leptokurtic ($\beta_2 > 3$), or mesokurtic ($\beta_2 = 3$), or platykurtic ($\beta_2 < 3$) depending on the ratio $\tau_0/\Delta t$.

Our group has studied the independence between log-returns and waiting times for stocks traded at the New York Stock Exchange in October 1999. A contingency-table analysis performed on general electric (GE) prices shows that the null hypothesis of independence can be rejected with a significance level of 1% [54]. We have also discussed the anomalous non-exponential behaviour of the unconditional waiting-time distribution between tick-by-tick trades both for future markets [55] and for stock markets [54,102]. Different waiting-time scales have been investigated in different markets by various authors. All these empirical analyses corroborate the waiting-time anomalous behaviour. A study on the waiting times in a contemporary FOREX exchange and in the XIXth century Irish stock market was presented by Sabatelli et al. [111]. They were able to fit the Irish data by means of a Mittag-Leffler function as our group did before in a paper on the waiting-time marginal distribution in the German-bund future market [55]. Kyungsik Kim and Seong-Min Yoon studied the tick dynamical behaviour of the bond futures in Korean Futures Exchange (KOFEX) market and found that the survival probability displays a stretched-exponential form [112]. Finally, Ivanov et al. [113] confirmed that a stretched-exponential fits well the survival distribution for NYSE stocks as we suggested in Ref. [54]. In order to stress the relevance of non-exponentially distributed waiting times, let this author recall that a power-law distribution has been detected by Kaizoji and Kaizoji in analysing the calm time interval of price changes in the Japanese market [114] and Barabasi has noticed that non-Poissonian processes characterize the activity of human beings [115,116]. In the case of financial markets, our group has confirmed that also waiting times between orders do not follow the exponential distribution and has offered an explanation in terms of variable human activity during trading days [102,117]. Finally, Muchnik and Solomon have recently developed a general simulation platform called NatLab in order to simulate agent-based asynchronous processes [118].

The results mentioned above can be further put into context and related to mainstream financial and mathematical finance research on price dynamics [119,120]. Recently, thanks to the availability of large high frequency and tick-by-tick data sets, there has been an increasing interest on the statistical properties of high-frequency financial data related to market microstructural properties [121–126]. High-frequency econometrics is also well established after research on autoregressive conditional duration models [127–130].

The remark that in high-frequency financial data not only returns but also waiting times between consecutive trades are random variables can be found in Ref. [131]. This remark is also present in a paper by Lo and MaKinley [132], but it can be traced at least to papers on the application of compound Poisson processes [77] and subordinated stochastic processes [61,62] to finance. Models of tick-by-tick financial data based on compound Poisson processes can also be found in the following references by Rydberg and Shephard [133–135].

4. Conclusions

This paper contains an outline of the theory of CTRWs as well as a discussion of their application to the theory of ruin for insurance companies, to Gibrat’s model for growth and inequality, and to price dynamics in
financial markets. Originally introduced in Physics as models for normal and anomalous self-diffusion, CTRWs turn out to be useful tools in many other fields of human knowledge.

In the scientific literature, instances and generalizations of CTRWs appear under different labels such as: compound Poisson processes, point processes with reward, pure jump processes with i.i.d. jumps. Some of these processes have been already in use in the ruin theory of insurance since the beginning of the XXth century, well before the paper by Montroll and Weiss [1].

The application to ruin theory is mature and it does not deserve more comments in this final section.

On the contrary, as for the application to growth and inequality, this is rather new and work by various authors is in progress. It is highly likely that the idea of growth shocks arriving at random times will foster further empirical research on firm growth.

As shown in Section 3.3, the statistical behaviour of returns sampled with a fixed time step can be rather different from the behaviour of tick-by-tick returns. In the language of firm growth this means that the statistical properties of the yearly growth rate can dramatically differ from the properties of the distribution of elementary shocks.

For what concerns financial applications of CTRWs there are several possible extensions of the work presented here [51].

First of all, one can abandon the hypothesis of i.i.d. log-returns and waiting times and consider various forms of dependence. In this case, it is no longer possible to exploit the nice properties of Laplace and Fourier transforms of convolutions, but, still, Monte Carlo simulations can provide hints on the behaviour of these processes in the diffusive limit.

A second possible extension is to include volumes as a third stochastic variable. This extension is straightforward, starting from a three-valued joint probability density.

A third desirable extension is to consider a multivariate rather than univariate model that includes correlations between time series.

The present author is eager to know progress in any direction by other independent research groups in the field of applications of CTRWs to Finance and Economics. This will be very useful to maintain this review up to date for future editions. He can be contacted at scalas@unipmn.it.

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