Market Dynamics as a Consequence of Local Complementarity and Global Substitutability in Agent’s Strategies

M. Bagella  
Economics Department  
University of Rome “Tor Vergata”

G. Susinno*  
Computational and Quantitative Finance  
Finance-and-Physics

R. Ciciretti  
Economics Department  
University of Rome “Tor Vergata”

February 8, 2005

Abstract

Empirical analysis of financial markets has shown number of stylized facts such as heavy tails or volatility “bursts” which are difficult to explain in terms of evolution of fundamental economic variables. Indeed the non-Gaussian, non-stable character of empirical distributions, such as excess demand or stock returns, demonstrate the weakness of any “independent agent” approach to model the real market. Starting from the existing literature on the characterization of the behavior of random economies with many interacting agents, we identify a set of microeconomic interaction rules which could help to explain the macroeconomic observed market behavior. Following the work of Bornholdt and extending the Brock and Durlauf work, we will consider interacting agents whose payoff exhibit both a strategic complementarity with their nearest neighbors actions and an eventual global substitutability with the global market state. In this set-up we reconstruct a price process related to the imbalance between buyers and sellers. Finally we investigate how the frustration resulting from the tendency of local imitation, with an additional coupling with the average state of the system reproduces main observed stylized facts of real financial markets. We show how in this framework even the largest crash may emerge as a natural intrinsic metastable dynamics of the system induced by a collective phenomena such as crowd effects or “herd” behavior.

FIRST DRAFT VERSION 1.0 September 1st, 2004
THIS DRAFT VERSION 2.0 December 7th, 2004
Do not redistribute or quote without the author’s authorization.
Usual disclaimer applies.

SUBMITTED TO: Applied Mathematical Finance

*Corresponding author. Email Address: susinno@finance-and-physics.org
1 Introduction

The standard approach of microeconomic theory, is to consider preferences of individual economic agents as fixed initial data. In other words, as stated by Koopmans in 1957 [1], an agent is not allowed “to indulge in a certain randomness in his responses to given circumstances”. But what if this stringent condition is relaxed? It is still possible to lay down theoretical results bearing some usefulness for the understanding of market dynamics? Indeed a careful analysis of the economic literature shows some essays in characterizing the behavior of random economies with many interacting agents. Hans Föllmer’s work in 1973 [2] is, with evidence, one of the first to tackle this problem by allowing random preferences of economic agents (as in Hildenbrand [3]) and assuming the probability law governing that randomness dependent on the agent’s environment. The theoretical framework of Föllmer, which bear a close relationship to models in statistical mechanics, has been further developed by Brock and Durlauf [15]. Assuming global rationality in the agent’s behavior, they develop a framework characterizing discrete decisions if agents experience private as well social utilities from their choices. Indeed they show the existence of multiple, locally stable equilibria of average behavior, when social utility effects are large enough in a non cooperative decision-making process. These results in Economic theory are extremely interesting both because they open a brand new field of investigation and because they show the potentiality of the synergy between Physics and Economics. Benefits may arise not only importing technicalities from “hard sciences” to social sciences but through a strong methodological interaction.

The interplay between Physics and Economy has produced interesting investigations of social interactions which are at the origin of market dynamics. Indeed number of empirical observations such as heavy tails or volatility “bursts” are difficult to explain in terms of evolution of fundamental economic variables. As reviewed in [6], the non-Gaussian, non-stable character of empirical distributions, as in excess demand or stock returns, demonstrate the weakness of any “independent agent” approach to model the real market. Indeed the dependence between agents seems to be an essential characteristics of the market structure. Market dynamics seems to emerge as a consequence of the interaction of a large number of agents. Moreover it will be interesting to give a mathematical characterization of panic events [7]. Indeed it seems that significative price fluctuations are not necessarily related to the arrival of informations or variations in fundamental economic variables [4, 5]. Therefore one is lead to think that extreme market movements may be caused by an intrinsic metastable dynamics of the system induced by a collective phenomena such as crowd effects or “herd” behavior.

Recently the literature on the subject has witnessed an increasing contribution from physicists. Indeed the same analytical and methodological tools that are applied to understand interacting particles dynamics can be relevant to understanding virtually any complex dynamical system including financial markets. Thus Economics has appeared to be a fertile field which has received a great deal of attention from physicists. This invasion is known as Econophysics. In 2003 Feigenbaum [21] wrote a nice and instructive review, accessible to both economists and physicists, to some of the theoretical approaches employed by physicists in this literature and compares them with methods used by pure economists.

In this work we study how stylized facts, deriving from empirical market observations, can be reconstructed by an extremely simplified model of interacting agents with binary choices. Our approach will be to start from the economic literature on the subject as reviewed by Brock and Durlauf on [15] and to perturb the analytical solvable equilibrium problem by allowing agents to exhibit in their payoff a local strategic complementarity and a global strategic substitutability as defined by Cooper and John in [16]. Indeed, following [11] we investigate how the frustration resulting from the tendency of local imitation, local herding [8, 9, 10, 6], with an additional coupling with the average state of the system reproduces main observations of real financial markets. Since in this case the system will exhibit a metastable dynamics we will analyze his macroscopic evolution through a numerical simulation assuming a simple communication structure based on ideas from statistical physics in which agents occupy the nodes of a n-periodic lattice and are influenced by the decisions of their nearest neighbors.

The paper is divided into eight sections (including introduction and conclusions). Section 2 is introduces the framework of a binary choice (buy/sell) model as in [15]. Since in the case of global rationality the model can be exactly solved we will discuss how single/multiple equilibria may appear from a random economy with many interacting agents. In Section 3, we extend the model to allow local reinforcement by neighbors imitation and a feed-back effect in terms of information about the global system status. In Section 4 we define a price process as function of the imbalance between buyers and sellers. In Section 5 we introduce the effect of exogenous information (bad/good news) on agent’s private utilities and show how a flow of positive news may allows the system to reach a higher level of macroscopic organization. Section 6 quickly describes the Metropolis algorithm used to simulate a synthetic market evolution compatible with the interacting agents model. In section 7 we discuss the results and compare them with market observations.
Let us first define the economic framework by defining agents, their payoff, and characterize their random idiosyncratic response to a given set of observables. As in [15] we consider that there are observables accessible both by the agent and the modeler, and observables accessible only by the agent. Therefore, following [15] we consider:

- A population of $N$ agents;
- The action $s_i$ of each agent $i = 1, \cdots, N$ belong to a binary choice set, i.e. $s_i \in \{-1; +1\}$;
- The agent’s choice is made to maximize a payoff $V$;
- The characteristics available for the choice are:
  - Observables to the modeler and agent $i$: $O_i$
  - Unobservables to the modeler but observables to the agent $\epsilon_i$

The $\epsilon_i$ are random shocks deriving from an idiosyncratic decision process of the agent $i$ given his available information set.

- The $\epsilon_i$ are considered as extreme value distributed such as:
  \[
  P(\epsilon_i(1) - \epsilon_i(-1) \leq x) = \frac{1}{1 + \exp(-\beta x)}
  \]

Setting aside the econometric practical desire to obtain the random utility term logistically distributed, this assumption can be considered as strong and universal. Indeed it means that the probability density function of agents’ actions $s_i = \{-1; +1\}$ are distributed according to a Maxwell-Boltzmann distribution function. Each configuration of the system has a probability given by the Boltzmann factor:

\[
p = \frac{1}{C} \exp(-\beta V)
\]  

where $C$ is a normalization constant, and the $\beta$ term can be interpreted as a market temperature describing the degree of randomness in the behavior of agents.

It is important to note that, in complex physical systems, the Maxwell-Boltzmann distribution is independent of the detailed specification of agent-agent interactions, as long as they exist. Therefore this fact endows the Maxwell-Boltzmann distribution with universality. As a consequence, even if it is almost impossible to determine the overall evolution of a particular agent, the macroscopic behavior of the system can be described from random interactions drawn from adequate probability distributions, even without the exact specification of the microscopic interactions.

- The individual decision process is defined by:
  \[
  \min_{\{s_i\}} - V(s_i, O_i, p_i(s_{-i}), \epsilon_i)
  \]
where $s_{-i}$ denotes the vector of all choices but $i$. Therefore $p_i(s_{-i})$ denote the individual’s beliefs concerning the choices of other agents, which is assumed independent of the realization of any $\epsilon_i$.

- Assume that the payoff function can be decomposed as:
  \[
  V(s_i, O_i, p_i(s_{-i}), \epsilon_i) = U(s_i, O_i, p_i(s_{-i})) + S(s_i, O_i, p_i(s_{-i})) + \epsilon_i
  \]
- Assume:
  \[
  S(s_i, O_i, p_i(s_{-i})) = -E_i \left\{ \sum_{i \neq j} \frac{J_{ij}}{2} (s_i - s_j)^2 \right\}
  = \sum_{i \neq j} J_{ij} (s_i E_i \{s_j\} - 1)
  \]
  as a social utility term.
Indeed, let
\[ U(s_i, O_i, p_i(s_{-i})) = h(s_i, O_i, p_i(s_{-i})) \cdot s_i + k_i = h_i \cdot s_i + k_i \]
as a personal utility term given the macroscopic behavior of the system seen by agent \( i \).

The probability that individual \( i \) makes the choice \( s_i \) is equal to the probability that the payoff associated to this choice exceed that of \( -s_i \), i.e.:
\[
P(s_i \mid O_i, p_i(s_{-i})) = P\left(h_i s_i + \sum_{i \neq j} J_{ij} s_i E_i\{s_j\} + \epsilon(s_i) < -h_i s_i - \sum_{i \neq j} J_{ij} s_i E_i\{s_j\} + \epsilon(-s_i)\right) \propto \exp\left(2\beta_i h_i s_i + 2 \sum_{i \neq j} \beta_i J_{ij} s_i E_i\{s_j\}\right) \tag{3}\]

Therefore a joint set of choices \( \vec{s} \) obeys to:
\[
P(\vec{s} \mid O_i, p_i(s_{-i})) \propto \prod_i \exp\left(2\beta_i h_i s_i + 2 \sum_{i \neq j} \beta_i J_{ij} s_i E_i\{s_j\}\right)
\]
And the rational expectation of the agent’s choice is given by:
\[
E(s_i) = \frac{\exp\left(2\beta_i h_i + 2 \sum_{i \neq j} \beta_i J_{ij} E_i\{s_j\}\right)}{\exp\left(2\beta_i h_i + 2 \sum_{i \neq j} \beta_i J_{ij} E_i\{s_j\}\right) + \exp\left(-2\beta_i h_i - 2 \sum_{i \neq j} \beta_i J_{ij} E_i\{s_j\}\right)}
\]
\[
= -1 \cdot \frac{\exp\left(-2\beta_i h_i - 2 \sum_{i \neq j} \beta_i J_{ij} E_i\{s_j\}\right) + \exp\left(-2\beta_i h_i - 2 \sum_{i \neq j} \beta_i J_{ij} E_i\{s_j\}\right)}{\exp\left(2\beta_i h_i + 2 \sum_{i \neq j} \beta_i J_{ij} E_i\{s_j\}\right) + \exp\left(-2\beta_i h_i - 2 \sum_{i \neq j} \beta_i J_{ij} E_i\{s_j\}\right)}
\]
\[
\equiv \tanh\left(2\beta_i h_i + 2 \sum_{i \neq j} \beta_i J_{ij} E_i\{s_j\}\right) \tag{4}\]

In the case when the agents all possess rational expectations, subjective expectations are replaced by their mathematical counterparts, i.e.,
\[
E_i(s_j) = E(s_j)
\]
we have:
\[
E(s_i) = \tanh\left(2\beta_i h_i + 2 \sum_{i \neq j} \beta_i J_{ij} E(s_j)\right) \tag{5}\]

and it is easy to verify that it admits at least one solution.

Indeed in the case of global rational expectation, the mathematical setting is simplified and \( E(s_i) \) can be considered as a measure of the average long range order of the system. Therefore, in the hypothesis of a global rational expectations, it is assumed that “there is no short range order apart from that which follows from long range order”. It is a well known result in statistical mechanics since it corresponds to the Bragg-Williams approximation [12] of the two dimensional Ising Model [13]. It accounts for the fact that if only long range interactions arising from local imitation are considered, then the model admits an analytical solution. It must be noted that in economics, if the axiom of global rationality holds, which means that one can safely neglect short range interactions i.e. there are no sizeable effects from local coalitions in the market behavior, the result of Eq. 5 is exact.

In this case, since Eq. 5 represent a continuous mapping of \([-1, 1]^N\) to \([-1, 1]^N\) (see [15]), by the Brouwer’s fixed point theorem, there is at least one self-consistent expectation for this binary choice model. Moreover, for a homogeneous system, i.e. \( N \) identical agents with identical observable characteristics \( h_i = h, \beta_i = \beta, J_{ij} = J \), any \( M \) is a self-consistent solution for the average choice if it solves:
\[
M = \tanh\left(2\beta h + 2\beta J N M\right)
\]
In this case, as shown in Fig. 1, there is a unique solution if \( 2\beta J N < 1 \) and three solutions for \( 2\beta J N > 1 \). Indeed, let \( \xi = 2\beta J N \), we have the following configurations:
\[ Y = X, \quad \text{or} \quad Y = \text{tanh} (\xi X), \quad 0 < \xi < 1 \]

1. \( h = 0 \):
   - For \( 0 < \xi < 1 \) the fundamental equilibrium \( M^0 = 0 \) is unique and stable.
   - For \( \xi > 1 \) the fundamental equilibrium \( M_0 = 0 \) becomes unstable, and two new equilibria appear: the bull market equilibrium \( M^+ \) and the bear market equilibrium \( M^- \) which are stable. At the bull (resp. bear) market equilibrium more than half of the agents are in the status “+1” (resp. “-1”).

2. \( h \neq 0 \):
   - For \( h > 0 \) and \( 0 < \xi < 1 \) Fig. 2 shows that there is a unique equilibrium for the system which is shifted to the bull market phase. By contrast, as \( h < 0 \), the system shifts to the bear market phase. Moreover the equilibrium is stable.
   - For \( \xi > 1 \) the system has two stable equilibria: \( m^+, m^- \) and one unstable: \( m^* \) if \( |\beta h| < \beta h^* = H_c \) where \( \beta h^* \) is determined by:
     \[ \cosh^2 \left( H_c \pm \sqrt{\xi (\xi - 1)} \right) = \xi \]

Finally if \( |\beta h| > H_c \) only one stable equilibrium remains.

3 Local complementarity and Global Substitutability

Starting from the previous model set-up we can generalize the construct to a more complex case. Indeed, assume that the system admits a structure of nearest-neighbor interacting agents, i.e. denote by \( n_i \) the number of neighbors of agent \( i, \forall i \in \{n\text{-dimensional periodic lattice}\} \), and:

\[
J_{ij} = \begin{cases} 
J_{ij} \neq 0 & \text{if } j \in n_i \\
0 & \text{otherwise.}
\end{cases}
\]

This term will account for the social interaction \( S(s_i, O_e, p_i(s_{-i})) \) and \( J_{ij} \geq 0 \) measures the strategic complementarity between individual choices and the expected choices of his neighbors.

Moreover we assume that, with respect to the average macroscopic state of the system, the personal utility of each agent may exhibit either a strategic complementarity (if they belong to the minority group) or a strategic substitutability (if they belong to the majority group). In such case, local interactions will tend to align the expectations of each agent while the interaction with the expected average global state of the system will push
$$M = \tanh(\beta h + \xi M)$$

$$Y = \tanh(h + \xi X), 0 < h < H_c$$

$$Y = \tanh(H_c + \xi X), H_c > 0$$

$$Y = \tanh(\xi X), h > H_c$$

$$\xi > 1$$

Figure 2: (a) Three equilibria for $\xi > 1$ and $h > 0$, (b) Three equilibria for $\xi > 1$ and $h < 0$, (c) equilibria for $\xi < 1$ and $h > 0$, and (d) equilibria for $\xi < 1$ and $h < 0$. 
agents in the minority to join the majority, and agents in the majority to join the minority. Therefore let us rewrite the personal utility as [11]:

\[
U(s_i, O_i, p_i(s_{-i})) = h_is_i + k_i = -\alpha_{ij}C(s_i)s_i \frac{1}{N} \sum_{j=1}^{N} E(s_j) + k_i = -\alpha_{ij}C(s_i)E(M) \cdot s_i + k_i;
\]

with \(\alpha_{ij} \geq 0\). Moreover we choose \(C(s_i)\) such that:

\[
C(s_i) = \begin{cases} 
+1 & \text{if } s_i \text{ belongs to the majority i.e.: } \text{sign}(s_i) = \text{sign}(M) \\
-1 & \text{Otherwise};
\end{cases}
\]

thus the payoff \(V\) can be rewritten as:

\[
V(s_i, O_i, p_i(s_{-i}), \epsilon_i) = -\alpha_{ij}C(s_i)E(M) \cdot s_i + \sum_{<ij>} J_{ij} s_i E_i(s_j) + \epsilon(s_i) + k_i
\]  \(6\)

where \(<ij>\) denotes the summation over the nearest-neighbors of agent \(i\). Again we find a strategic complementarity [16] between individual choices and the expected choices of others:

\[
\frac{\partial^2 V(s_i, O_i, p_i(s_{-i}), \epsilon_i)}{\partial s_i \partial E_i(s_j)} = J_{ij} \geq 0
\]

but now we have as well a term in the private utility which can exhibit either strategic complementarity or substitutability [16] between individual choices and the expected average status of the system \(E(M)\):

\[
\frac{\partial^2 V(s_i, O_i, p_i(s_{-i}), \epsilon_i)}{\partial s_i \partial E(M)} = -\alpha_{ij}C(s_i)
\]

Finally we assume that the best expectation of agent \(i\) about the state of his neighbor \(j\) is the observed state \(s_j\), i.e.:

\[
E_i(s_j) \equiv s_j
\]

and:

\[
M = \frac{1}{N} \sum_{j=1}^{N} s_j
\]

Given the assumed distribution of \(\epsilon(s_i)\) we deduce that:

\[
P(s_i = +1) = \frac{1}{1 + \exp(-2\beta \left[ \sum_{<ij>} J_{ij} s_j - \alpha_{ij}C(s_i)M \right])}
\]

\[
P(s_i = -1) = 1 - P(s_i = +1);
\]  \(7\)

The framework outlined above can be used as a starting toy model to investigate the market dynamic emerging from the interaction of many agents. Indeed we consider a network of \(N\) agents. Each agent \(i\), located in a node of a network, is modeled by a simple state or spin. We will start from a simple two state model i.e. \(s_i = \pm 1\), \(s_i = 1\) represent a buyer and \(s_i = -1\) a seller [11]. Eventually, as in [10] we may think to introduce a “0” state to identify locally inactive traders. We impose to the network a periodicity condition such as agents on the boundaries of the lattice are connected nord-south and east-west. Indeed this corresponds to a torus as shown in Fig. 3. Each trader is supposed to interact with his nearest neighbors with a constant interaction energy \(J_{ij} = J\) if sites \(i, j\) are directly connected and \(J_{ij} = 0\) otherwise. Moreover each agent has access to the global average state of the system. Therefore in this model (Eq. 6) there are two kind of actors: those who tend to mimic the majority (\(C_i = -1\)) and those who try to take advantage being in the minority (\(C_j = +1\)). There is always the probability (Eq. 7) of a switch from majority to minority players but this can happen only at a cost. The transition rule is given by the fact that a trader which is in the majority group will try to switch in the minority one in order to take the maximum advantage from a future fashion movement (buy low sell high). On the other hand a minority agent can be unsatisfied from his returns and may eventually decide to join the crowd. In that case, as exposed in [11], a majority agent \(i\) will always act with a strategy spin \(C_i = +1\), while a minority agent \(j\) will act with a strategy spin \(C_j = -1\). If this applies continuously the payoff of agent \(i\) can be written as:

\[
V(s_i, O_i, p_i(s_{-i}), \epsilon_i) = -\alpha_{ij} \mid E(M) \mid s_i + \sum_{<ij>} J_{ij} s_i E_i(s_j) + \epsilon(s_i) + k_i
\]  \(8\)
in that case $\alpha_{i,j}$ can be seen as the average coupling impact of the expected global state $\bar{X}$ on the agent $i$. This simple model give rise to a non trivial dynamics. Indeed a high level of imbalance between buyers (+1 state) and sellers (-1 state) generate an metastable system with global structures created by the local herding. Those structures may be followed by a sudden order disruption and rapid rearrangement typical of over-critical systems. Indeed the characteristics that are stylized in this toy market model are:

- Local reinforcement by neighbors imitation.
- Global feed-back effect in terms of information about the global system status.
- Minority/Majority players with a non vanishing probability to jump from a group to the other. This is an important aspect since a trend follower desire will be to exit (resp enter) the market before the crowd. Therefore a successful trader will be in the minority when changing his position and in the majority during inactivity.
- A idiosyncratic behavior in the decision process seen as a thermal noise. This noise is introduced to take into account the bounded rationality of an agent. Indeed a real market player cannot act following a fully rational expectations otherwise, for an agent to have rational beliefs, he has to have beliefs about what all the other agents are going to do. They in turn have to have beliefs about what he is going to do, which means he also has to have beliefs about their beliefs about what he is going to do. And they need to have beliefs about these beliefs and so on, leading to an infinitely complicated problem that no economist, let alone the average investor, knows how to solve [21]. Therefore in the decision process a market actor must introduce a subjective estimate on the way to act, i.e. a noise with respect to the global system and his neighbors state.

The dependence of the payoff with the average state of the system produce a feedback mechanism which may introduce a Self Organized Critical behavior [19] and intermittency in the dynamics. This construction tends to mimic, in a simple and extremely stylized manner, the influence of an aggregate price process on individual investors which in turn take their positions according both to the feeling of their nearest neighbors and their own idiosyncratic believes on the macroscopic evolution of the system.

The effect of the macroscopic price process on the individuals may act in different ways according to the agent strategy and behavior. Indeed the agent may be in a state of noisy trader aligning his/her actions to those of the majority. Conversely the agent may decide a contrarian position deciding to act doing what the minority
does. Moreover, the role of the heat bath dynamics is to model the relative strength in the decision process between the idiosyncratic feeling of an agent (which can be seen as a pure thermal noise) and the local/global state of the system.

This model specification, assuming also a lattice structure will allow us to simulate the dynamics of the system. It remains to define how prices may be created in such toy market.

**Figure 4: Metropolis algorithm [18]**

4 The price process

As proposed by Kaizoji et al [22] we may think of two groups of market players and a clearing system mechanism. Fundamentalists will produce an order size $X^F$ proportional to the misalignment of the price $p(t)$ with respect to a “fundamental” price $p^*(t)$ therefore:

$$X^F(t) \propto \ln (p^*(t)) - \ln (p(t))$$

On the other hand, noise traders orders $X^N$ are proportional to the average imbalance measured as $M(t)$, where $N$ is the number of interactive traders. Therefore:

$$X^N(t) \propto M(t)$$

The clearing mechanism imposes that:

$$X^N(t) + X^F(t) = \ln (p^*(t)) - \ln (p(t)) + \lambda M(t) \equiv 0$$

Assuming for simplicity $p^*(t) = p = 1$ one obtains:

$$r(t-1,t) = \ln \left( \frac{p(t)}{p(t-1)} \right) = \lambda [M(t) - M(t-1)]$$

Therefore, as a first approximation we just assume the increments of the global average imbalance as a proxy for the prices log-returns in the artificial market.
In that sense it is may worth a note the recent proposition made by Cross et al [23]. In their recent preprint they propose a similar price mechanism based on the average imbalance between buyers and sellers, i.e.:

\[ \sigma(t) = M(t) \]
\[ p(t) = p(t - 1) \cdot \exp \left( \sqrt{\tau} \Delta W(t) + k \Delta \sigma(t) \right) \]

where \( W(t) \) represents the creation of new, uncorrelated and globally available, information over a time period \( \tau \). The variable \( \Delta \sigma(t) = \sigma(t) - \sigma(t - 1) \) is the most recent change in market sentiment and the constant \( k > 0 \) determines the average effect that a single agent has on the market price. The larger the value of \( k \), the more the market price is influenced by internal market dynamics as opposed to the generation of new market information. Indeed this approach is little bit weaker since there is no general global mechanism turning the macroscopic state into a microscopic contribution for the agent’s action flip mechanism.

5 Exogenous Information

The model allow us to introduce an additional source of randomness such as an exogenous source of global information \( I_e \).

\[
V (s_i, O_i, p_i(s_{-i}), \epsilon_i) = -\alpha_{ij} s_i (\mid E (M_j) \mid + I_e) + \sum_{<ij>} J_{ij} s_i E_i (s_j) + \epsilon (s_i) + k_i
\]  

(9)

In that case, we may allow for two kind of information:

- **Bad news**: \( I_e(t) < 0 \) will force players to listen more to the global evolution decreasing eventually the characteristic correlation length of the system. Tends to destabilize the system. However if the system is in a stable configuration (no imbalance: \( \mid M(t) \mid \approx 0 \)) bad news can be absorbed by the system without crashing.
• **Good news:** $I_e(t) > 0$ will make players more self confident in listening their mates and neighbors. Enthusiasm driven by good news may eventually push to higher level of organization. The higher limit of metastable under-critical configuration, produced by good news, can eventually amplify the amplitude of the reversion process induced by the spontaneous order breaking: a bigger crash.

The previous observations can intuitively be derived from the equilibrium analysis we made by the inspection of Eq. 5. Indeed a large imbalance in the system will produce a higher value of $|E(M)|$ and eventually this could push the personal utility term above the critical value $H_c$ disrupting the multiple equilibria setup. Conversely a flow of good informations will contribute to moderate the influence of $|E(M)|$, allowing the system to reach a higher level of macroscopic organization. This can be seen in the numerical tests obtained applying the simulation method exposed in the next section. The system in its initial status is reported in Fig. 5, after $N$ time steps the system is in an ordered phase and in Fig. 6. An increase of the order of the system at $N+1$ steps may destroy the internal organization as in Fig. 7.

### 6 Simulation method

A frequently used approach to simulate the behavior of a system is the Metropolis algorithm [18]. Again, since a two dimensional Ising model consists of $N$ sites where a particular microstate of the lattice is specified by the set of variables $\tilde{s} = \{s_1, \cdots, s_N\}$. The Metropolis Monte Carlo method is a computational approach for generating a set of configurations of the system $\tilde{s}_1, \cdots, \tilde{s}_M$ according to a given probability distribution $p(\tilde{s})$ (see eq. 1). As shown in Fig. 4 the Metropolis algorithm can be defined by the following steps:

1. Pick a random initial configuration $\tilde{s}_0$;
2. Pick a trial configuration $\tilde{s}_t$, usually close to $\tilde{s}_i$, and compute a probability ratio $r = p(\tilde{s}_t)/p(\tilde{s}_i)$. Pick a random number $z \in [0, 1]$. Then make $s_{i+1} = \tilde{s}_t$ if $z < r$ or let $s_{i+1} = s_i$ otherwise.
3. Go to step 2 replacing $\tilde{s}_i$ with $s_{i+1}$.
7 First Results

At a first glance, we observe in the real-Time Metropolis simulations a number of interesting features which share a lot with real market phenomenology. In Fig. 8, the evolution of the daily prices for the S&P500 is reported. The time windows spans from 1982 to 2004.

It is interesting to note that the crash induced by the 2001 September 11th terrorist attack has a relative amplitude which is 5 times smaller than the jump of the black Monday 1987, where the market had a drawback of almost 20% without any airplane crashing near by wall street (Fig. 8). Indeed the terrorist attack took place in a period when the inflationary bubble of the late nineties did already start to resorb [25]. In the light of the model presented here we may imagine that the maximum metastable state has been reached by the year 2000, when the dot com bubble start to collapse, and at the time of the Twin Towers terrorist attack the system had already reached a state stable enough to absorb the shock. Indeed it seems that significant price fluctuations may not necessarily be related to the arrival of informations or variations in fundamental economic variables as reported in [4, 5]. As a comparison in Fig. 9 we show the results of a simulation. We took a grid of 128 X 128 agents with $\alpha = 40$ and a idiosynchratic noise equivalent to a temperature $T = 1.0 K < T_c$. We choose a temperature $T$ smaller than the critical temperature $T_c$ since we want a herding process to take place. For temperatures higher than $T_c$ the idiosynchratic noise would be high enough to dominate the dynamics. We want agents to listen to their neighbors and not only to their subjective decision process. But listening to the neighbors moves the system towards a high level of imbalance making riskier to remain with the majority of the crowd. As the order of the system increase people start to get more and more influenced by the global information waiting for the decision to jump out of the majority. The interesting observation is that such system may naturally evolve to a metastable correlated configuration where the smallest perturbation may triggers a massive migration from the ordered metastable state to a less ordered but stable one. The system is also able to reproduce the persistency observed in the autocorrelation of the absolute returns (Fig. 11) while, as in the market signal Fig. 10, there is no autocorrelation for the returns. We observe also that there is an analogy between the absolute value of the imbalance $|M(t)|$ and the volumes in the market, market instability should coincide with high volumes exchanges.
Finally we observe that such a simplified model give rise also to a distribution of the returns with fat tails. Indeed, given a high threshold $u$, the distribution of excess values of $|r(t)|$ over threshold $u$ is defined by:

$$F_u(y) = \Pr(|r(t)| - u \leq y \mid |r(t)| \geq u) = \frac{F(y + u) - F(u)}{1 - F(u)}$$

which represents the probability that the value of $|r(t)|$ exceeds the threshold $u$ by at most an amount $y$ given that $|r(t)|$ exceeds the threshold $u$. Following a theorem of Belkema and de Haan [24], for sufficiently large threshold $u$ the distribution function of the excess may be approximated by the generalized Pareto distribution (GPD) such that, as the threshold gets large, the excess distribution $F_u(y)$ converges to the GPD, which is:

$$G(x) = \begin{cases} 1 - \left(1 - \frac{\xi \eta}{\eta}ight)^{-1/\xi}, & \text{if } \xi \neq 0 \\ 1 - e^{-x/\beta}, & \text{if } \xi = 0 \end{cases}$$

The parameter $1/\xi$ is called the tail index. For $\xi > 0$ the distribution is heavy tailed and $E[x^t]$ is infinite for $t \geq 1/\xi$. In most market time series $\xi$ lies between 0.25 and 0.5.

In our example, with a cut-off of 1.05 for the financial and synthetic Ising time series, we observe: $\xi = 0.24 \pm 0.06$ and $\xi = 0.33 \pm 0.06$ respectively.

### 7.1 Are Traders Mad?

If in one side we live in a infinite rational world we may be tempted to reject the approach presented here. After all how a rational human being can act in such a way to contribute to burn a fortune? Indeed it can happens just
because infinite rationality when thinking in a multi-period, multi-agents economy cannot be applied efficiently
to each single agent. Each decision is always taken subjectively conditioned to the available information. This
margin of subjectivity let open the probability to be a better/worse player than the colleagues. To win the game
each player must be in the minority side just before the crowd is attracted by this side, and with the majority
during the fashion rump-up. As the game goes on, the neighbors imitation tend to be more and more dangerous
since all the players are looking for the right moment to jump out from the majority to join the minority side.
Just think about gain consolidation or stop loss. During the herding phase, all the agents have tendency to
imitate each other, the more the imitation orders the system the higher is the risk of a rapid inversion. At a
given stage each majority market player wants to satisfy his appetite for future gains but has also to deal with
his increasing fear. In critical conditions the system is so stressed that even the smallest perturbation to the
system may trigger a catastrophic reaction. In that way each agent can behave rationally given his available
information but the aggregate behavior can produce a catastrophic event giving rise to a quite irrational but
natural crash.

Therefore it is tempting to think that extreme market movements may be an intrinsic metastable dynamics
of the system induced by a collective phenomena such as crowd effects or “herd” behavior.
8 Conclusions

The Ising Model is one of the simplest fundamental models of statistical mechanics. It is used to describe phenomena such as magnets; liquid/gas coexistence; alloys of two metals and, probably, bearish/bullish market players. These systems are described by local spins with values $s_i = \pm 1$ corresponding to a binary choice set. The choices can be interpreted as up/down spins, atom A or atom B in an alloy, buy/sell, etc. These variables generally describe the status of an element in a site $i$ of a lattice (Fig. 3). The Ising model, even in its simplest form, is extremely powerful in describing the Order/Disorder phase transitions.

The macroscopic behavior of a system depends on its lattice structure but only a very limited set of elementary configurations admit an analytical solution. However even the most stylized description of the system may be useful to the investigation. The frequent use of two dimensional lattices in Ising models, as a first approximation, is often the first step toward the understanding of the origins of observed macroscopic dynamics.

Models akin to statistical mechanics have only recently been introduced in microeconomic Theory by Föllmer [2] and the Economic literature on this subject is still quite scarce. As in [15] we show how under the assumption of perfect, global rationality the macroscopic equilibria deriving from local strategic complementarity can explain a fashion rump-up at the origin of a collective behavior responsible for bull/bear market phases, and how information can generate a bull $\leftrightarrow$ bear transition. The physical interpretation of global rationality hypothesis is that it assumes that “there is no short range order apart from that which follows from long range order” corresponding to the well known Bragg-Williams [12] approximation in statistical mechanics.

By considering a two-dimensional $n$-periodic square lattice, we perturb the theoretical construct of Brock and Durlauf by allowing both local imitation, by strategic complementarity between agents, and global substitutability between agents and aggregated market status. In that case we obtain a system which admits intermittency in his dynamics without reaching a defined equilibrium. Indeed if one considers the organization induced by local imitation (e.g. a market index, an aggregated price process,...), as an additional source of information available to the agent, a metastable dynamics is recovered which admits the same stylized facts observed from empirical analysis of market evolution.

In our configuration each agent sits on a node of a grid with four neighbor. Therefore there is no hierarchy or cluster structure on the connections of each player. By changing the network configuration, moving towards more complex connections, the possibility to compute analytically the characteristic parameters of the system (such as the critical temperature $T_c$ or $\beta_c$) is almost lost. Therefore one has to rely on numerical simulations in order to investigate the dynamical properties of the system.

In 1998 Cont and Bouchaud [6] proposed a simple model where a random communication structure between agents may give rise to the stylized facts observed in empirical studies of high frequency market data. From an empirical viewpoint Bonanno et al [14], analyzed the topological characterization of the minimal spanning tree [20] obtained by considering the price returns correlations in stock markets. In this case, they find that the empirical tree has features of a complex network that cannot be reproduced, even as a first approximation, by a random market model and by the one-factor model. It is tempting to consider a tantamount relationship between the connection structure they obtain, comparing a large correlation matrices of stocks returns, and the distribution of the connections between agents. In this case we could try to map the minimal connected tree
obtained by analyzing the correlated evolution of market returns of a large number of assets into an agents’ network of “spin agents” and compare the model’s behavior with observed data. We plan to tackle this problem in the near future.
References


